A FORMULA REPRESENTING MAGNETIC BEREZIN TRANSFORMS AS FUNCTIONS OF THE LAPLACIAN ON $\mathbb{C}^n$

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ABSTRACT. we give a formula that express magnetic Berezin transforms associated with generalized Bargmann-Fock spaces as a functions of the Euclidean Laplacian on $\mathbb{C}^n$.

1 INTRODUCTION

The Berezin transform was introduced by Berezin [1] for certain classical symetric domains in $\mathbb{C}^n$. This transform links the Berezin symbols and the symbols for Toeplitz operators. It is present in the study of the correspondence principle. The formula representing the Berezin transform as function of the Laplace-Beltrami operator plays a key role in the Berezin quantization [1].

This transform can be defined as follows. Consider a domain $D \subseteq \mathbb{C}^n$ and a Borel measure on D. Let $\mathcal{H}$ be a closed subspace of $L^2(D, d\mu)$ consisting of continuous functions and we assume that $\mathcal{H}$ has a reproducing kernel $K(.,.)$. Then, The Berezin symbol $\sigma(A)$ of a bounded linear operator A on $\mathcal{H}$ is the function on D given by $\sigma(A)(z) = \langle Ae_z, e_z\rangle$, where $e_z(.) = K(z,z)^{-\frac{1}{2}}K(.,z) \in \mathcal{H}$. For each $\varphi \in L^\infty(D)$, the Toeplitz operator T_φ with symbol φ is the operator on $\mathcal{H}$ given by $T_\varphi[f] = P(\varphi f)$, $f \in \mathcal{H}$ where P is the orthogonal projection from $L^2(D, d\mu)$ into $\mathcal{H}$. The Berezin transform associated to $\mathcal{H}$ is, by definition, the positive self-adjoint operator $\sigma(T)$, which turns out to be a bounded operator on $L^2(D, d\mu)$, where $d\mu_K = K(z,z)d\mu(z)$.

Now, based on the consideration that the Berezin transform can be defined provided that there is a given closed subspace L^2 which possesses a reproducing kernel, we are here concerned with the eigenspaces

$$A_m^2(\mathbb{C}^n) = \left\{\psi \in L^2(\mathbb{C}^n, e^{-|z|^2}d\mu), \quad \widetilde{\Delta}\psi = \epsilon_m\psi\right\} \tag{1.1}$$

of the second order differential operator

$$\widetilde{\Delta} = -\sum_{j=1}^{n}\frac{\partial^2}{\partial z_j\partial\bar{z}_j} + \sum_{j=1}^{n}\bar{z}_j\frac{\partial}{\partial\bar{z}_j}, \tag{1.2}$$

corresponding to eigenvalues $\epsilon_m = m$, $m = 0,1,2,\ldots$. Here $d\mu$ is the Lebesgue measure on $\mathbb{C}^n$. The operator $\widetilde{\Delta}$ constitutes (in suitable units and up to an additive constant), in $L^2(\mathbb{C}^n, e^{-|z|^2}d\mu)$, a realization of the Schrödinger operator with uniform magnetic field in $\mathbb{C}^n$. Theses eigenspaces are reproducing kernel Hilbert spaces with reproducing kernels given by ([2]):

$$K_m(z,w) := \pi^{-n}e^{\langle z,w\rangle}L_m^{(n-1)}\left(|z-w|^2\right), w, z \in \mathbb{C}^n, \tag{1.3}$$

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where $L_k^{(\alpha)}(x)$ is the Laguerre polynomial [3, p. 239].

Actually, by [2] it is known that the eigenspace $A_0^2(\mathbb{C}^n)$ corresponding to $m = 0$ coincides with Bargmann-Fock $\mathcal{F}(\mathbb{C}^n)$ space of holomorphic functions that are $e^{-|z|^2}d\mu$- square integrable, while for $m \neq 0$, the spaces $A_m^2(\mathbb{C}^n)$ which can be viewed as kernel spaces of the hypoelliptic differential operator $\left(\widetilde{\Delta} - m\right)$, consist of non holomorphic functions, These spaces are called generalized Bargmann-Fock spaces.

Note also, for $m = 0$, the Berezin transform, denoted B_0, associated with the Bargmann-Fock space $\mathcal{F}(\mathbb{C}^n)$ turns out to be given by a convolution product over the group $\mathbb{C}^n = \mathbb{R}^{2n}$ as

$$B_0[\phi](z) := \left(\pi^{-n}e^{-|w|^2} * \phi\right)(z), \qquad \phi \in L^2(\mathbb{C}^n, d\mu).$$

Furthermore, it can be expressed as a function of the Euclidean Laplacian on $\mathbb{C}^n$ as $B_0 = e^{\frac{1}{4}\Delta_{\mathbb{C}^n}}$ [4].

In this paper, we extend to each eigenspace $A_m^2(\mathbb{C}^n)$ the notion of Berezin transform by considering the transform defined via the following convolution product over $\mathbb{C}^n$ as

$$B_m[\phi](z) := \frac{\pi^{-n}m!}{(n)_m}\left(e^{-|w|^2}\left(L_m^{(n-1)}(|w|^2)\right)^2 * \phi\right)(z), \phi \in L^2(\mathbb{C}^n, d\mu), \tag{1.4}$$

and we prove that this transform can also be expressed as a function of the Laplacian $\Delta_{\mathbb{C}^n}$ as:

$$B_m := \frac{1}{(n)_m}e^{\frac{1}{4}\Delta_{\mathbb{C}^n}}\sum_{k=0}^{m}\frac{(n-1)_k\,(m-k)!}{k!}\left(\frac{\Delta_{\mathbb{C}^n}}{4}\right)^k L_{m-k}^{(k)}\left(\frac{\Delta_{\mathbb{C}^n}}{4}\right)L_{m-k}^{(n-1+k)}\left(\frac{\Delta_{\mathbb{C}^n}}{4}\right), \tag{1.5}$$

where $(\alpha)_j = \alpha(\alpha+1)\cdots(\alpha+j-1)$ denotes the Pochhammer symbol.

This paper is organized as follows. In Section 2 we recall same needed facts on the generalized Bargmann-Fock spaces. In Section 3, we apply the formalism of the Berezin transform so as to extend this notion to each generalized Bargmann-Fock spaces. In Section 4, we give a formula that represents the extended Berezin transform as a function of the Laplacian in the Euclidean complex n-space .

2 The Schrödinger operator with magnetic field on $\mathbb{C}^n$.

The motion of charged particle in a constant uniform magnetic field in $\mathbb{R}^{2n}$ is described (in suitable units and up to additive constant) by the Schrödinger operator:

$$H_B := -\frac{1}{4}\sum_{j=1}^{n}\left(\partial_{x_j} + By_j\right)^2 + \left(\partial_{y_j} - iBx_j\right)^2 - \frac{n}{2} \tag{2.1}$$

acting on $L^2\left(\mathbb{R}^{2n}, d\mu\right)$, $B > 0$ is a constant proportional to the magnetic field strength. We identify the Euclidean space $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way. The operator H_B in equation (2.1) can be represented by the operator

$$\widetilde{H}_B = e^{\frac{1}{2}B|z|^2}H_Be^{-\frac{1}{2}B|z|^2}. \tag{2.2}$$

According to equation (2.2), an arbitrary state ϕ of $L^2(\mathbb{R}^{2n}, d\mu)$ is represented by the function $Q[\phi]$ of $L^2(\mathbb{C}^n, e^{-|z|^2}d\mu)$defined by

$$Q[\phi](z) := e^{\frac{1}{2}|z|^2}\phi(z), z \in \mathbb{C}^n. \tag{2.3}$$

The unitary map Q in (2.3) is called ground state transformation. For $B = 1$, the explicit expression for the operator in equation (2.2) turns out to be given by the operator $\widetilde{\Delta}$ introduced in equation (1.2). The Latter is considered with $C_0^\infty(\mathbb{C}^n)$ as its regular domain in the Hilbert space $L^2(\mathbb{C}^n, e^{-|z|^2}d\mu)$ of $e^{-|z|^2}d\mu$-square integrable functions $\varphi : \mathbb{C}^n \to \mathbb{C}$.

We let $P_m : L^2(\mathbb{C}^n, e^{-|z|^2}d\mu) \to A_m^2(\mathbb{C}^n)$ denote the orthogonal projection operator onto the eigenspace $A_m^2(\mathbb{C}^n)$ as defined in (1.1). A basis elements of this space can be written explicitly in terms of the Laguerre polynomials $L_k^{(\alpha)}(x)$ and the polynomials $h_{p,q}^j(z,\overline{z})$ whose are the restriction to the unit sphere S^{2n-1} of harmonic homogeneous polynomials of bidegree (p,q) [5, p. 253]. Precisely, the following set of the functions

$$\Psi_{j,p,q}^m(z) = \left(\frac{2(m-q)!}{\Gamma(n+m+p)}\right)^{\frac{1}{2}} L_{m-q}^{(n+p+q-1)}\left(|z|^2\right) h_{p,q}^j(z,\overline{z}) \tag{2.4}$$

constitutes an orthonormal basis of $A_m^2(\mathbb{C}^n)$, for varying $p = 0,1,2,\ldots; q = 0,1,\ldots,m$ and $j = 1,\ldots,d_{p,q}^n$ with $d_{p,q}^n = \dim \mathcal{H}_{p,q}\left(S^{2n-1}\right)$, where $\mathcal{H}_{p,q}\left(S^{2n-1}\right)$ is finite dimensional vector space spanned by the above harmonic polynomials $h_{p,q}^j$. The above basis can be used to obtain the reproducing kernel of the Hilbert space $A_m^2(\mathbb{C}^n)$ by following general theory [6]. Actually, as mentioned in section 1, this kernel is of the form

$$K_m(z,w) = \pi^{-n} e^{\langle z,w\rangle} L_m^{(n-1)}\left(|z-w|^2\right), z,w \in \mathbb{C}^n \tag{2.5}$$

For more informations on the spectral properties of the operator $\widetilde{\Delta}$ and its eigenspaces $A_m^2(\mathbb{C}^n)$ we refer to [7].

Remarque 2.1 Note that, for $m = 0$, the kernel $K_0(z,w) = \pi^{-n}e^{\langle z,w\rangle}$ coincides with the reproducing kernel of Bargmann-Fock $\mathcal{F}(\mathbb{C}^n)$.

3 Magnetic Berezin Transforms

According to the formalism described in Section 1, we take as domain $D = \mathbb{C}^n$ the whole complex space. For a bounded operator A on $\mathcal{H} := A_m^2(\mathbb{C}^n) \subset L^2(\mathbb{C}^n, e^{-|z|^2}d\mu)$, the Berezin symbol of A isdefined by

$$\sigma_m(A) := \langle Ae_{z,m}, e_{z,m}\rangle_{\mathcal{H}} \tag{3.1}$$

where $e_{z,m}(.) := (K_m(z,z))^{-\frac{1}{2}} K_m(z,.)$ denotes the normalized reproducing kernel according to (2.5) of $A_m^2(\mathbb{C}^n)$ with evaluation at $z \in \mathbb{C}^n$, precisely,

$$e_{z,m}(w) = \pi^{\frac{n}{2}} \left(\frac{m!}{(n)_m}\right)^{-\frac{1}{2}} e^{-\frac{|z|^2}{2}} K_m(z,w), \quad w \in \mathbb{C}^n. \tag{3.2}$$

For a bounded function on $\mathbb{C}^n$, the Toeplitz operator T_φ is the operator $T_\varphi(h) = P_m(\varphi h), h \in L^2(\mathbb{C}^n, e^{-|z|^2}d\mu)$. the Berezin transform of the function φ is defined to be the Berezin symbol $\sigma_m\left(T_\varphi\right)$. That is,

$$B_m[\varphi](z) := \sigma_m\left(T_\varphi\right)(z) = \left\langle T_\varphi\left(e_{z,m}\right), e_{z,m}\right\rangle_{\mathcal{H}}, z \in \mathbb{C}^n \tag{3.3}$$

Explicitly, this transform reads

$$B_m[\varphi](z) = \frac{m!}{(n)_m \pi^n} \int_{\mathbb{C}^n} e^{-|z-w|^2} \left(L_m^{(n-1)}\left(|z-w|^2\right)\right)^2 \varphi(w) d\mu(w), \tag{3.4}$$

where $\varphi \in L^\infty(\mathbb{C}^n)$.

As mentioned in the introduction, is easy to see from (3.4) that the transform B_m, can written as a convolution operator as

$$B_m[\varphi] = b_m * \varphi, \quad \varphi \in L^2(\mathbb{C}^n, d\mu) \tag{3.5}$$

where

$$b_m(z) = \frac{m!}{(n)_m \pi^n} e^{-|z|^2} \left(L_m^{(n-1)}(|z|^2) \right)^2, z \in \mathbb{C}^n. \tag{3.6}$$

Is not difficult to see that the function b_m belongs to $L^1(\mathbb{C}^n)$ by making use of the orthogonality relation of Laguerre polynomials [3, p. 241]. Indeed, $\|b_m\|_{L^1(\mathbb{C}^n)} = \pi^{-n}$. Then, applying Hausdorff-Young inequality to $b_m * \varphi$ unable us to write that

$$\|B_m[\varphi]\|_{L^2(\mathbb{C}^n)} \leq \pi^{-n} \|\varphi\|_{L^2(\mathbb{C}^n)} \tag{3.7}$$

which means that $B_m : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ is a bounded operator and such as we will be dealing with.

Remark 3.1. Note that, a decomposition of the action of B_m on the product of radial functions with spherical harmonics have been discussed in [8].

4 Berezin transform and the Euclidean Laplacian

In this section, we shall express the transform B_m as function of the Euclidean Laplacian of $\mathbb{C}^n$. For this, we first state the following:

Proposition 4.1. *Let $m \in \mathbb{Z}_+$, then there exists a function f_m such that $B_m = f_m(\Delta_{\mathbb{C}^n})$, with $f_m(\lambda) = e^{-\frac{\lambda}{4}} P_m(\lambda)$, $P_m(.)$ being a polynomial function.*

Proof. Since the transform B_m can be written in view of (3.5) as a convolution over $\mathbb{C}^n$ as $B_m[\varphi] = b_m * \varphi$, $\varphi \in L^2(\mathbb{C}^n)$, the function b_m being defined in (3.6), then by the general theory ([9, p. 200]) we can write that:

$$B_m = \widehat{b}_m \left(\frac{1}{i} \nabla \right) \tag{4.1}$$

where

$$\widehat{b_m}(\xi) = \int_{\mathbb{C}^n} e^{-i(\xi|w)} b_m(w) d\mu(w) \tag{4.2}$$

is the Fourier transform of the function b_m. Here, $(\xi \mid w)$ denotes the real scalar product in $\mathbb{C}^n = \mathbb{R}^{2n}$ and ∇ is the gradient operator on $\mathbb{R}^{2n}$. Using the expression of the Laguerre polynomial ([3, p. 240]):

$$L_m^{(n-1)}(x) = \sum_{k=0}^{m} (-1)^k \binom{m+n-1}{m-k} \frac{x^k}{k!}, \tag{4.3}$$

then, from (**??**) we can write

$$L_m^{(n-1)}\left(|w|^2\right) = \sum_{k=0}^{m} c_k |w|^{2k}, \quad c_k = \frac{(-1)^k}{k!} \binom{m+n-1}{m-k}. \tag{4.4}$$

Inserting in (4.2) the expression of the Laguerre polynomial given in (4.4), we obtain

$$\begin{aligned}\widehat{b}_m(\xi) &= \frac{m!}{\pi^n (n)_m}\sum_{k=0}^{m}\sum_{j=0}^{m} c_j c_k \int_{\mathbb{C}^n} e^{-|w|^2} e^{-i(\xi|w)}\, |w|^{2(j+k)}\, d\mu(w) \\ &= \frac{m!}{\pi^n (n)_m}\sum_{k=0}^{m}\sum_{j=0}^{m} c_j c_k \left(\Delta_\xi\right)^{j+k}\left(\int_{\mathbb{C}^n} e^{-|w|^2} e^{-i(\xi|w)} d\mu(w)\right) \end{aligned} \tag{4.5}$$

where $\Delta_\xi = 4\sum\limits_{j=1}^{n} \frac{\partial^2}{\partial \xi_j \partial \overline{\xi}_j}$ is the Laplacian in terms of the variable ξ.

The last integral is recognized as the Gaussian integral ([10, p. 153]):

$$\int_{\mathbb{C}^n} e^{-|w|^2} e^{-i(\xi|w)} d\mu(w) = \pi^n e^{-\frac{|\xi|^2}{4}} \tag{4.6}$$

Thus, making use of (4.6), equation (4.5) reads

$$\widehat{b}_m(\xi) = \frac{m!}{(n)_m}\sum_{k=0}^{m}\sum_{j=0}^{m} c_j c_k \Delta_\xi^{j+k}\left(e^{-\frac{|\xi|^2}{4}}\right). \tag{4.7}$$

Now, the last term in (4.7) can be expressed as

$$\Delta_\xi^{j+k}\left(e^{-\frac{|\xi|^2}{4}}\right) = P_{j,k}\left(|\xi|^2\right) e^{-\frac{|\xi|^2}{4}}. \tag{4.8}$$

where $P_{j,k}(.)$ is polynomial function. Therefore, equation (4.7) becomes

$$\widehat{b}_m(\xi) = \frac{m!}{(n)_m}\sum_{k=0}^{m}\sum_{j=0}^{m} c_j c_k P_{j,k}\left(|\xi|^2\right) e^{-\frac{|\xi|^2}{4}} = e^{-\frac{|\xi|^2}{4}} P_m(|\xi|^2) \tag{4.9}$$

Finally, replacing ξ by $\frac{1}{i}\nabla$, we arrive at announced statement of Proposition 1. This ends the proof.

Theorem 4.2. *Let $m \in \mathbb{Z}_+$. Then, the Berezin transform B_m can be expressed in terms of the Laplacian $\Delta_{\mathbb{C}^n}$ as*

$$B_m = \frac{1}{(n)_m}\exp\left(\frac{\Delta_{\mathbb{C}^n}}{4}\right)\sum_{k=0}^{m}\frac{(n-1)_k (m-k)!}{k!}\left(\frac{\Delta_{\mathbb{C}^n}}{4}\right)^k L_{m-k}^{(k)}\left(\frac{-\Delta_{\mathbb{C}^n}}{4}\right) L_{m-k}^{(n-1+k)}\left(\frac{-\Delta_{\mathbb{C}^n}}{4}\right) \tag{4.10}$$

Using of Proposition 1 together with general theory of the function of self-adjoint operator ([11, p. 338]), we can link the Berezin transform B_m with the spectral family $\{E_\lambda, \lambda > 0\}$ associated to $-\Delta_{\mathbb{C}^n}$ as

$$\langle B_m \varphi, \psi\rangle_{L^2(\mathbb{C}^n)} = \langle f_m\left(-\Delta_{\mathbb{C}^n}\right)\varphi, \psi\rangle_{L^2(\mathbb{C}^n)} = \int_0^{+\infty} f_m(\lambda), d\left\langle E_\lambda[\varphi], \psi\right\rangle_{L^2(\mathbb{C}^n)} \tag{4.11}$$

$\varphi, \psi \in L^2(\mathbb{C}^n)$ and E_λ is the well known spectral projector given by ([9, 9p. 202]):

$$E_\lambda \varphi(z) = \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n}\int_{\mathbb{C}^n} |z-w|^{-n} J_n\left(\lambda^{\frac{1}{2}}|z-w|\right)\varphi(w) d\mu(w), \tag{4.12}$$

$J_\nu(.)$ being the Bessel function of order ν ([3, p. 65]).

Using an integration by part in the Stieltjes' sense, we obtain from (4.11)

$$\langle B_m\varphi,\psi\rangle_{L^2(\mathbb{C}^n)} = -\int_0^{+\infty}\frac{df_m(\lambda)}{d\lambda}\langle E_\lambda\varphi,\psi\rangle_{L^2(\mathbb{C}^n)}\,d\lambda \tag{4.13}$$

$$= \left\langle\int_0^{+\infty}-\frac{df_m(\lambda)}{d\lambda}E_\lambda[\varphi]d\lambda,\psi\right\rangle_{L^2(\mathbb{C}^n)},\forall\psi\in L^2(\mathbb{C}^n,d\mu) \tag{4.14}$$

This relation (4.14) implies that

$$B_m[\varphi] = \int_0^{+\infty}-\frac{df_m(\lambda)}{d\lambda}E_\lambda[\varphi]d\lambda,\quad \varphi\in L^2(\mathbb{C}^n,d\mu) \tag{4.15}$$

which can also be written, in view of (4.12), as:

$$B_m[\varphi](z) = \frac{1}{(2\pi)^n}\int_0^{\infty}\frac{-df_m(\lambda)}{d\lambda}\lambda^{\frac{n}{2}}\int_{\mathbb{C}^n}\frac{J_n\left(\lambda^{\frac{1}{2}}|z-w|\right)}{|z-w|^n}\varphi(w)d\mu(w)d\lambda \tag{4.16}$$

$$= \int_{\mathbb{C}^n}\frac{-1}{(2\pi)^n}\left(\int_0^{+\infty}\lambda^{\frac{n}{2}}\frac{df_m(\lambda)}{d\lambda}\frac{J_n\left(\lambda^{\frac{1}{2}}|z-w|\right)}{|z-w|^n}d\lambda\right)\varphi(w)\,d\mu(w). \tag{4.17}$$

On the other hand, recalling the expression of B_m given in (3.4) as:

$$B_m[\varphi](z) = \frac{m!}{(n)_m\pi^n}\int_{\mathbb{C}^n}e^{-|z-w|^2}\left(L_m^{(n-1)}(|z-w|^2)\right)^2\varphi(w)d\mu(w). \tag{4.18}$$

We are lead to consider the following equality:

$$\frac{-1}{(2\pi)^n}\int_0^{+\infty}\left(\lambda^{\frac{n}{2}}\frac{df_m(\lambda)}{d\lambda}\frac{J_n\left(\lambda^{\frac{1}{2}}|z-w|\right)}{|z-w|^n}\right)d\lambda = \frac{m!}{(n)_m\pi^n}e^{-|z-w|^2}\left(L_m^{(n-1)}(|z-w|^2)\right)^2 \tag{4.19}$$

The equation (4.19) can be also written as

$$\int_0^{+\infty}\lambda^{\frac{n}{2}}g_m(\lambda)J_n\left(\lambda^{\frac{1}{2}}x\right)d\lambda = -C_{n,m}x^ne^{-x^2}\left(L_m^{(n-1)}(x^2)\right)^2 \tag{4.20}$$

where $g_m(\lambda) = \frac{df_m(t)}{dt}\,|_{t=\lambda}$, $x = |z-w|$ and $C_{n,m} = \frac{2^n m!}{(n)_m}$.

Changing the variable of integration by writing $\lambda = s^2$, we get from (4.20)

$$\int_0^{+\infty}s^ng_m(s^2)sJ_n(sx)ds = \frac{1}{2}C_{n,m}x^ne^{-x^2}\left(L_m^{(n-1)}(x^2)\right)^2. \tag{4.21}$$

The left hand side of equation (4.21) can be presented as Hankel transform as

$$\mathcal{H}_n\left[s\mapsto s^ng_m(s^2)\right](x) = \frac{1}{2}C_{n,m}x^ne^{-x^2}\left(L_m^{(n-1)}(x^2)\right)^2 \tag{4.22}$$

where $\mathcal{H}_n$ is defined by (see [12, p. 67]):

$$\mathcal{H}_n[u(s)](x) = \int_0^{+\infty}u(s)sJ_n(sx)ds. \tag{4.23}$$

and satisfies the involution property

$$\int_0^{+\infty}\left(\int_0^{+\infty}u(x)J_\nu(tx)\,xdx\right)J_\nu(ts)tdt = u(s) \tag{4.24}$$

which holds for every continuous function u on $]0,+\infty[$ with $\int_0^{+\infty}x^{\frac{1}{2}}|u(x)|dx<\infty$).

Making use of the involution property (4.24) for $\nu = n$, $u(s) = s^n g_m(s^2)$, the equation (4.22) becomes

$$s^n g_m\left(s^2\right) = -\frac{1}{2}C_{n,m}\int_0^{+\infty} x^{n+1}e^{-x^2}\left(L_m^{(n-1)}(x^2)\right)^2 J_n(xs)dx. \tag{4.25}$$

This, implies that the function $g_m(\lambda)$ is of the form

$$g_m(\lambda) = -\frac{1}{2}C_{n,m}\int_0^{+\infty} x^{n+1}e^{-x^2}\left(L_m^{(n-1)}(x^2)\right)^2 \lambda^{-\frac{n}{2}} J_n\left(x\lambda^{\frac{1}{2}}\right)dx = \frac{df_m(\lambda)}{d\lambda}. \tag{4.26}$$

Let $t > 0$, then by an integration over the interval $]t, +\infty[$, we have

$$\lim_{\lambda\to+\infty} f_m(\lambda) - f_m(t) = -\frac{1}{2}C_{n,m}\int_0^{+\infty} x^{n+1}e^{-x^2}\left(L_m^{(n-1)}(x)\right)^2 \int_t^{+\infty}\lambda^{-\frac{n}{2}} J_n\left(x\lambda^{\frac{1}{2}}\right)d\lambda dx. \tag{4.27}$$

Taking into account the form of the function f_m in the proposition 1, we have

$$\lim_{\lambda\to+\infty} f_m(\lambda) = \lim_{\lambda\to+\infty} e^{-\frac{\lambda}{4}}P_m(\lambda) = 0. \tag{4.28}$$

Then, we obtain:

$$f_m(t) = \frac{1}{2}C_{n,m}\int_0^{+\infty} x^{n+1}e^{-x^2}\left(L_m^{(n-1)}(x^2)\right)^2 \int_t^{+\infty}\lambda^{-\frac{n}{2}} J_n\left(x\lambda^{\frac{1}{2}}\right)d\lambda dx. \tag{4.29}$$

Observe that the substitution $\lambda = ts$ transforms the last integral in (4.29) to

$$\int_t^{+\infty}\lambda^{-\frac{n}{2}} J_n\left(x\lambda^{\frac{1}{2}}\right)d\lambda = t^{1-\frac{n}{2}}\int_1^{+\infty} s^{-\frac{n}{2}} J_n\left(\left(xt^{\frac{1}{2}}\right)\sqrt{s}\right)ds, \tag{4.30}$$

and by making use of formula ([13, p. 691])

$$\int_1^{+\infty}\rho^{-\frac{n}{2}}(\rho-1)^{\mu-1}J_\nu(a\sqrt{\rho})d\rho = \Gamma(\mu)2^\mu a^{-\mu}J_{\nu-1}(a), \tag{4.31}$$

$a > 0$ and $0 < \Re(\mu) < \frac{1}{2}\Re(\nu) + \frac{3}{4}$ for $a = xt^{\frac{1}{2}}$, $\mu = 1$ and $\nu = n$, we obtain

$$\int_t^{+\infty}\lambda^{-\frac{n}{2}} J_n\left(x\lambda^{\frac{1}{2}}\right)d\lambda = 2x^{-1}(\sqrt{t})^{1-n}J_{n-1}(x\sqrt{t}). \tag{4.32}$$

Consequently, the function $f_m(t)$ expressed by the formula (4.29), becomes:

$$f_m(t) = C_{n,m}t^{-\frac{n}{2}+\frac{1}{2}}\int_0^{+\infty} x^n e^{-x^2}J_{n-1}(x\sqrt{t})\left(L_m^{n-1}(x^2)\right)^2 dx \tag{4.33}$$

To calculate the integral (4.33), we will discuss two cases: $n = 1$ and $n \geq 2$.

For $n = 1$, we make use the formula ([13, p. 812])

$$\begin{aligned}&\int_0^{+\infty} x^{\nu+1}e^{-\alpha x^2}L_p^{(\nu-\sigma)}\left(\alpha x^2\right)L_q^{(\sigma)}(\alpha x^2)J_\nu(xy)dx\\ &\quad= \frac{(-1)^{p+q}}{2\alpha}\left(\frac{y}{2\alpha}\right)^\nu \exp\left(-\frac{y^2}{4\alpha}\right)L_p^{(\sigma-p+q)}\left(\frac{y^2}{4\alpha}\right)L_q^{(\nu-\sigma+p-q)}\left(\frac{y^2}{4\alpha}\right)\end{aligned} \tag{4.34}$$

for $y = \sqrt{t}, \alpha = 1, \nu = 0$ and $\sigma = 0$. Then, we obtain that

$$f_m(t) = C_{1,m}\int_0^{+\infty} x^n e^{-x^2}J_0(x\sqrt{t})L_m^{(0)}(x^2)L_m^{(0)}(x^2)dx = e^{-\frac{t}{4}}\left(L_m^{(0)}\left(\frac{t}{4}\right)\right)^2. \tag{4.35}$$

So, we get

$$B_m = \exp\left(\frac{1}{4}\Delta_{\mathbb{C}}\right)\left(L_m^{(0)}\left(-\frac{1}{4}\Delta_{\mathbb{C}}\right)\right)^2. \tag{4.36}$$

For $n \geq 2$, we first make use of the identity ([3, p. 249])

$$L_p^{(\alpha)}(y) = \sum_{k=0}^{p} \frac{(\alpha-\beta)_k}{k!} L_{k-p}^{(\beta)}(y) \tag{4.37}$$

for $p = m, \alpha = n-1, \beta = 0$ and $y = x^2$. Then, the equation (4.33) becomes

$$f_m(t) = C_{n,m} t^{\frac{1-n}{2}} \sum_{k=0}^{m} \frac{(n-1)_k}{k!} \int_0^{+\infty} x^n e^{-x^2} L_m^{(n-1)}(x^2) L_{m-k}^{(0)}(x^2) J_{n-1}(x\sqrt{t}) dx. \tag{4.38}$$

To calculate the integral in the equation (4.38), which is denoted

$$\mathcal{I}_k := \int_0^{+\infty} x^n e^{-x^2} L_m^{(n-1)}(x^2) L_{m-k}^{(0)}(x^2) J_{n-1}(x\sqrt{t}) dx, \tag{4.39}$$

we return back to formula (4.34) and use it for $\nu = n-1, \sigma = 0, p = m, q = m-k$ and $y = \sqrt{t}$. We obtain that

$$\mathcal{I}_k = \frac{1}{2^n} t^{\frac{n-1}{2}} \exp\left(\frac{-t}{4}\right) L_m^{(-k)}\left(\frac{t}{4}\right) L_{m-k}^{(n+k-1)}\left(\frac{t}{4}\right). \tag{4.40}$$

Next, making use of the identity ([3, p. 240]),

$$L_p^{(-k)}(x) = (-x)^k \frac{(p-k)!}{p!} L_{p-k}^{(k)}(x) \tag{4.41}$$

for $p = m$ and $x = \frac{t}{4}$, the integral $\mathcal{I}_k$ becomes

$$\mathcal{I}_k = \frac{1}{2^n} \frac{(m-k)!}{m!} \left(-\frac{t}{4}\right)^k t^{\frac{n-1}{2}} \exp\left(\frac{-t}{4}\right) L_{m-k}^{(k)}\left(\frac{t}{4}\right) L_{m-k}^{(n+k-1)}\left(\frac{t}{4}\right) \tag{4.42}$$

Then, we get that

$$\begin{aligned} f_m(t) =& \frac{1}{(n)_m} \exp\left(\frac{-t}{4}\right) \\ & \times \sum_{k=0}^{m} \frac{(n-1)_k (m-k)!}{k!} \left(-\frac{t}{4}\right)^k L_{m-k}^{(k)}\left(\frac{t}{4}\right) L_{m-k}^{(n-1+k)}\left(\frac{t}{4}\right). \end{aligned} \tag{4.43}$$

Finally, we can write that

$$\begin{aligned} B_m =& \frac{1}{(n)_m} \exp\left(\frac{1}{4}\Delta_{\mathbb{C}^n}\right) \\ & \times \sum_{k=0}^{m} \frac{(n-1)_k (m-k)!}{k!} \left(\frac{\Delta_{\mathbb{C}^n}}{4}\right)^k L_{m-k}^{(k)}\left(\frac{-\Delta_{\mathbb{C}^n}}{4}\right) L_{m-k}^{(n-1+k)}\left(\frac{-\Delta_{\mathbb{C}^n}}{4}\right) \end{aligned} \tag{4.44}$$

We should note that the expression (4.44) enable us to rederive (4.36) when replacing $n = 1$ with the convention $(0)_0 = 1$. This help us to summarize the discussion in one form as in the statement of the theorem.

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